LARGE SUPERUNIVERSAL METRIC SPACES

BY

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ABSTRACT

For every uncountable cardinal κ define a metric space S to be *κ-superuniversal* iff for every metric space U of cardinality κ, every partial isometry into S from a subset of U of cardinality less than κ can be extended to all of U. We prove that any such space must have cardinality at least $2^{\bar{\kappa}} = \Sigma_{\lambda<\kappa} 2^{\lambda}$, and for each regular uncountable cardinal κ, we construct a κ-superuniversal metric space of cardinality $2^{\bar{\kappa}}$, It is proved that there is a unique κ-superuniversal metric space of cardinality κ iff $2^{\bar{\kappa}} = \kappa$. Several decomposition theorems are also proved, e.g., every κ-superuniversal space contains a family of $2^{\bar{\kappa}}$ disjoint κ-superuniversal subspaces. Finally, we consider some applications to more general topological spaces, to graph theory, and to category theory, and we conclude with a list of open problems.

1. Introduction

In 1910 Maurice Fréchet [5, pp. 161–162] constructed a metric space of cardinality $2^{\aleph_0}$ which is universal with respect to all separable metric spaces, and in 1925 Paul Urysohn [16, 17] constructed a separable such universal space. Then in 1940 Waclaw Sierpiński [14] announced and later proved [15] that for each cardinal κ such that $2^{\bar{\kappa}} = \kappa$† there exists a metric space of cardinality κ which is universal with respect to all metric spaces of cardinality κ. Recently, Charles Joiner [8] simplified Urysohn's construction (and found a new homogeneity property connected with the space) and suggested that it be generalized to larger spaces. He also pointed out that Urysohn's space has the property that every isometry from a finite subset of the space into the space can be extended to an isometry from the entire space onto itself and noted that this property could be

† We define $2^{\bar{\kappa}} = \Sigma_{\lambda<\kappa} 2^{\lambda}$.

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generalized. In fact, this had already been done for certain cardinals by Michael Morley and Robert Vaught [11] in 1962 in a much more general setting. They observed that metric spaces form a Jónsson class [9, 10]. Bjarni Jónsson had proven general theorems about such classes which in the case of metric spaces yielded the existence and uniqueness of universal homogeneous metric spaces of cardinality κ, where κ is any uncountable regular cardinal satisfying $2^{\bar{\kappa}} = \kappa$, and the existence of universal metric spaces of cardinality $2^{\bar{\kappa}}$ for all regular uncountable cardinals.

However, when one deals with spaces which are strictly larger than the spaces with respect to which they are universal, the resulting homogeneity property is very weak; it allows us to extend isometries from "small" subsets of the space in question into itself to isometries over larger subspaces, but it does not allow us to extend these to isometries over the entire space. For this reason, the present author was led to look upon properties of this type not as weak homogeneity properties but rather as strong universality properties. This turns out to be almost equivalent to Isidore Fleischer's notion of C injectivity [4].

Thus for any uncountable cardinal κ, we define a metric space $\boldsymbol{S}$ to be **κ-superuniversal** iff, for every metric space $\boldsymbol{T}$ of cardinality at most κ, every isometry from some subset of $\boldsymbol{T}$ of cardinality less than κ into $\boldsymbol{S}$ can be extended to an isometry from all of $\boldsymbol{T}$ into $\boldsymbol{S}$. We prove that every κ-superuniversal metric space has cardinality at least $2^{\bar{\kappa}}$, and for regular uncountable cardinals we construct κ-superuniversal metric spaces of just this cardinality. It is also proved that all κ-superuniversal spaces of cardinality $2^{\bar{\kappa}}$ are isometric iff $2^{\bar{\kappa}} = \kappa$, and a variety of decomposition characterization theorems are considered. A generalization to proper classes and its resulting category theoretical formulation is also treated and it is shown that while κ-superuniversality can be extended to bounded metric spaces, it cannot be extended much further without leading to non-Hausdorff superuniversal spaces. Finally, an application to graph theory is presented.

The outline of this paper is as follows: In Section 2 the notation used throughout the paper is introduced; we define and briefly discuss some properties such as regularity etc., of cardinals which are required later, and we carry out the construction of the spaces which we later prove to have desirable superuniversality properties. Next, in Section 3, κ-superuniversality is studied in detail especially with respect to regular cardinals. In Section 4, a somewhat weaker concept is discussed which we call weak κ-superuniversality and which seems to be more appropriate for the study of singular cardinals. Finally, in Section 5, various generalizations are considered and in Section 6, some open problems are stated.

We wish at this point to express our gratitude to Charles Joiner for suggesting this topic, for making available to us a prepublication copy of [8], and for many stimulating discussions on this subject. We also wish to thank W. Wistar Comfort for bringing the works of Morley and Vaught, Jónsson, Sierpiński, and Fleischer to our attention and to thank Frank Harary for directing us to the work of Rado.

2. Notation and the spaces $H_\kappa^\prec$

For each infinite cardinal κ and certain well orderings of a certain set $\mathscr{F}$, we construct a metric space $H_\kappa^\prec$. Then in later sections, if will be shown that these spaces have the superuniversality properties of interest to us and are usually the smallest such spaces.

As has become customary, we identify cardinals with initial ordinals; that is the cardinality of a set A, which is denoted by $|A|$, is defined to be the smallest ordinal which can be mapped onto it. It should be noted that in making this definition the axiom of choice is assumed. We shall continue to assume this axiom throughout the paper.

We need a certain amount of notation. For any function f, any sets A and B, and any cardinal κ denote the domain of f by $\mathrm{dm}(f)$ the range of f by $\mathrm{rn}(f)$, the smallest ordinal α such that $\mathrm{dm}(f) \subseteq \alpha$ by $\mathrm{bn}(f)$, the function f restricted to the set A by $f \upharpoonright A$, the set $\{a \in A: a \notin B\}$ by $A - B$, the set of functions from A into B by ${}^AB$, the cardinality of ${}^AB$ by B^A, the power set of A by $\mathscr{P}(A)$, the set $\{C \subseteq A: |C| < \kappa\}$ by $\mathscr{P}_\kappa(A)$, the set $\{C \subseteq A: |C| \leqq \kappa\}$ by $\mathscr{P}_\kappa^+(A)$, the smallest cardinal greater than κ by κ^+, and $\Sigma_{\lambda<\kappa} 2^\lambda$ by $2^{\bar{\kappa}}$. If A has some ordering $\prec$ associated with it (in particular, if A is an ordinal), then we use $\mathscr{B}(A)$ to denote the set of bounded subsets of A, i.e., the set $\{C \subseteq A: \exists a \in A\ (c \in C \to c \prec a)\}$, and we use $\mathscr{B}_\kappa(A)$ and $\mathscr{B}_\kappa^+(A)$ to denote $\mathscr{P}_\kappa(A) \cap \mathscr{B}(A)$ and $\mathscr{P}_\kappa^+(A) \cap \mathscr{B}(A)$ respectively.

If $\boldsymbol{S} = \langle S, \mu \rangle$ and $\boldsymbol{T} = \langle T, \nu \rangle$ are any two metric spaces, then denote the set of isometries from S into T by $\mathscr{I}(S,T)$, the set of isometries from some subset of S into T by $\mathscr{S}\mathscr{I}(S,T)$, and the set of isometries from members of $\mathscr{P}_\kappa(S)$ or $\mathscr{P}_\kappa^+(S)$ into T by $\mathscr{S}\mathscr{I}_\kappa(S,T)$ and $\mathscr{S}\mathscr{I}_\kappa^+(S,T)$ respectively.

We reserve R to denote the real numbers, R^+ to denote the positive real numbers, $\boldsymbol{R}$ and $\boldsymbol{R}^+$ to denote the metric or topological spaces generated by R and R^+, and $<$ to denote the natural ordering on R or the natural ordering among the ordinals (precisely which, will always be clear from the context).

We frequently need to deal with the notions of regularity, cofinality, etc., of cardinals. A cardinal κ is said to be a **successor** cardinal iff it is equal to λ^+ for some cardinal λ and to be a **limit** cardinal otherwise. The **cofinality** of a cardinal κ, which we shall denote by $\mathrm{cf}(\kappa)$, is defined to be the smallest cardinal λ such that

$$\mathscr{P}_\lambda^+(\kappa) \neq \mathscr{B}_\lambda^+(\kappa),$$

i.e., such that κ contains an unbounded subset of cardinality λ. Equivalently, the cofinality of κ may be defined to be the smallest cardinal λ such that there exists a family $\{A_\alpha : \alpha < \lambda\}$ of λ sets each of cardinality strictly less than κ whose union $\bigcup_{\alpha<\lambda} A_\lambda$ has cardinality κ. We then define a cardinal κ to be **regular** iff

$$\mathrm{cf}(\kappa) = \kappa, \text{ or equivalently,}$$
$$\mathscr{P}_\kappa(\kappa) = \mathscr{B}_\kappa(\kappa)$$

and to be **singular** otherwise. It is well known that every successor cardinal is regular and that it is consistent with the axioms of Zermelo-Fraenkel set theory including choice (which we hereafter refer to as ZFC) that all uncountable regular cardinals be successor cardinals. A limit cardinal which is regular is generally referred to as an **inaccessible** cardinal.

Throughout this paper, we shall be interested in knowing when we can add a new point to a metric space with specified distances to at least some of the points already there. Thus if $\langle U, \mu \rangle$ is any metric space and f is any function from some subset $S \subseteq U$ into R^+, then we define f to be **consistent** iff

$$|f(s) - f(t)| \leqq \mu(s,t) \leqq f(s) + f(t) \qquad \text{for all } s, t \in S;$$

to be **superfluous at u** iff it is consistent and

$$\inf(\{f(s) + \mu(s,u) : s \in S\}) = 0;$$

and to be **superfluous** iff it is superfluous at some point u.

Intuitively, we think of f as consistent if it defines a set of distances which do not violate the triangle inequality and superfluous if it defines a set of distances which is uniquely satisfied by a point already in the space. More formally, we have:

2.1 LEMMA. *If $U = \langle U, \mu \rangle$ is any metric space and f is any function from some subset $S \subseteq U$ into R^+, then*:

1. *If for some point* $u \in U$

$$f(s) = \mu(u,s) \qquad \textit{for all } s \in S,$$

then f *is consistent.*

2. *If* f *is superfluous at some point* $u \in U$, *then*:
   a. $\mu(u,v) = \inf(\{f(s) + \mu(s,v) : s \in S\})$ *for all* $v \in U$,
   b. $\mu(u,s) = f(s)$ *for all* $s \in S$.

3. *If* f *is consistent but not superfluous and* v *is any point not in* U, *then* U *can be extended to a metric space* $\langle U \cup \{v\}, \nu \rangle$ *in which*:
   a. $\nu(v,u) = \inf(\{f(s) + \mu(s,u) : s \in S\})$ *for all* $u \in U$,
   b. $\nu(v,s) = f(s)$ *for all* $s \in S$.

PROOF. Part 1 is an immediate consequence of the triangle inequality. To prove 2.a, we note that from the triangle inequality and the consistency of f we have

$$\mu(u,v) \leqq \mu(u,t) + \mu(t,s) + \mu(s,v) \leqq \mu(u,t) + (f(t) + f(s)) + \mu(s,v)$$
$$= (\mu(u,t) + f(t)) + (\mu(s,v) + f(s)) \qquad \text{for all } s,t \in S.$$

Now taking the infimum over $t \in S$ of both sides and using the fact that f is superfluous, we obtain

$$\mu(u,v) \leqq f(s) + \mu(s,v) \qquad \text{for all } s \in S.$$

Hence

$$\mu(u,v) \leqq \inf(\{f(s) + \mu(s,v) : s \in S\}).$$

On the other hand, we note that

$$\mu(s,v) \leqq \mu(u,v) + \mu(u,s) \qquad \text{for all } s \in S,$$

so

$$f(s) + \mu(s,v) \leqq \mu(u,v) + f(s) + \mu(u,s) \text{ for all } s \in S.$$

Again taking the infimum over $s \in S$ and using the fact that f is superfluous at u, we have

$$\inf(\{f(s) + \mu(s,v)\}) \leqq \mu(u,v).$$

To prove 2.b, we note that the only way it could fail would be if for some $t \in S$, we had

$$f(t) + \mu(t,s) < f(s).$$

But this implies

$$\mu(t,s) < f(s) - f(t),$$

which violates the consistency of f.

Finally, the proof of 3 requires only a check of the triangle inequality. ■

Our construction of $H_\kappa^{\prec}$ consists of choosing a set and looking at all functions over it with domain of cardinality less than κ and range contained in R^+. Because we examine each such function separately, we need a set which has the same cardinality as the set of appropriate functions over it. The smallest such set turns out to be one of cardinality $2^{\bar\kappa}$ and for simplicity we use $2^{\bar\kappa}$ itself. It also turns out that when $2^{\bar\kappa}$ is singular, it is not always sufficient to require that the domains be bounded in cardinality; they must also be bounded in order.

Thus for each uncountable cardinal κ, let $\mathscr{F}_\kappa$ be the set of functions f into R^+ such that

$$\mathrm{dm}(f) \in \mathscr{B}_\kappa(2^{\bar\kappa}).$$

We note that if $\mathrm{cf}(2^{\bar\kappa}) \geqq \kappa$ (which is always the case if κ is regular), then we may replace $\mathscr{B}_\kappa(2^{\bar\kappa})$ by $\mathscr{P}_\kappa(2^{\bar\kappa})$.

2.2 Lemma *For every uncountable cardinal κ, the set $\mathscr{F}_\kappa$ has cardinality* $2^{\bar\kappa}$

Proof. Clearly $2^{\bar\kappa} \leqq |\mathscr{F}_\kappa|$. For each $A \in \mathscr{B}_\kappa(2^{\bar\kappa})$ we have

$$|\{f \in \mathscr{F}_\kappa : \mathrm{dm}(f) = A\}| = (R^+)^A = (2^{\aleph_0})^A \leqq 2^{\aleph_0 \cdot |A|} \leqq 2^{\bar\kappa}.$$

Thus we need only show that $|\mathscr{B}_\kappa(2^{\bar\kappa})| \leqq 2^{\bar\kappa}$. In particular, it is sufficient to show that for each $\alpha < 2^{\bar\kappa}$, we have $|\mathscr{P}_\kappa(\alpha)| \leqq 2^{\bar\kappa}$. But for any $\alpha < 2^{\bar\kappa}$ there exists a cardinal $\lambda < \kappa$ such that $\alpha < 2^\lambda$ so we have

$$|\mathscr{P}_\kappa(\alpha)| \leqq |\mathscr{P}_\kappa(2^\lambda)| \leqq \sum_{\gamma<\kappa} |\mathscr{P}_\gamma^+(2^\lambda)| \leqq \sum_{\gamma<\kappa} (2^\lambda)^\gamma \leqq \sum_{\gamma<\kappa} 2^{\bar\kappa} \leqq \kappa \cdot 2^{\bar\kappa} = 2^{\bar\kappa}. \quad ■$$

We can now carry out the construction of the spaces which we eventually use as examples of superuniversal spaces. Define a well ordering $\prec$ of $\mathscr{F}_\kappa$ to be **admissible** iff $\mathscr{F}_\kappa$ has order type $2^{\bar\kappa}$ under it; thus under an admissible ordering, we may think of $\mathscr{F}_\kappa$ as $\{f_\alpha : \alpha < 2^{\bar\kappa}\}$. Then for each uncountable cardinal κ and each admissible well ordering $\prec$ of $\mathscr{F}_\kappa$, we define a metric space

$$H_\kappa^{\prec} = \langle H_\kappa, \mu^{\prec} \rangle.$$

For H_κ, we use the set $2^{\bar\kappa}$ although, for easier readability, we frequently continue to denote it by H_κ; we define $\mu^{\prec}$ by transfinite induction using members of $\mathscr{F}_\kappa$.

Thus suppose we have already defined $\mu^{\prec}$ on α (i.e., we have defined $\mu^{\prec}(\beta,\gamma)$

for all $\beta,\gamma < \alpha$) and we wish to extend $\mu^{\prec}$ to the point α itself (i.e., we wish to define $\mu(\alpha,\beta)$ for all $\beta < \alpha$). Let δ be the least ordinal such that $\mathrm{dm}(f_\delta) \subseteq \alpha, f_\delta$ is consistent but not superfluous over $\langle \alpha, \mu^{\prec} \rangle$, and f_δ has not been used in a previous step of the construction. Denote this function f_δ by f^α and, using 2.1.3, define:

$$\mu^{\prec}(\alpha,\beta) = \inf(\{f^\alpha(d) + \mu^{\prec}(d,\beta) : d \in \mathrm{dm}(f^\alpha)\}) \quad \text{for all } \beta \in \alpha.$$

It remains to be proven that there always exists such a δ. This can be done using the methods of 3.2 ,but we shall not do so here. It is sufficient to note that if such a δ does not exist, we can always obtain one by dropping the condition that f_δ has not been used, without adversely affecting the construction.

The crucial property of $H_\kappa^{\prec}$ is:

2.3 Theorem. *If $f \in \mathscr{F}_\kappa$ is consistent and $f = f_\delta$ under $\prec$, then there exists a point $\alpha \in H_\kappa$ such that:*

a. $|\alpha| \leqq |\max(\delta, \mathrm{bn}(f), \aleph_0)|$, *and*

b. $\mu^{\prec}(\alpha,\eta) = f(\eta)$ *for all $\eta \in \mathrm{dm}(f)$.*

Proof. Let $\sigma = \max(\delta, \mathrm{bn}(f), \aleph_0)$. At each stage in the construction after σ, it will be possible to consider f. But there will be at most δ functions which may be considered after σ and yet before $f_\delta = f$; so if γ is the stage at which f is actually considered, we have $\gamma \leqq \sigma + \delta$. At this time, f will become f^γ unless there is a point below γ with respect to which it is superfluous. If there is such a point β, set $\alpha = \beta$; otherwise set $\alpha = \gamma$. In the first case, part b follows from 2.1.2, while in the second it follows immediately from the definition of $\mu^{\prec}$. In either case, part a is satisfied because

$$|\alpha| \leqq |\gamma| \leqq |\sigma + \delta| \leqq |\sigma + \sigma| = |\sigma|. \qquad \blacksquare$$

3. Superuniversality and regular cardinals

Using our new notation we repeat the definition of κ-superuniversality.

A metric space $\langle S, \mu \rangle$ is κ-**superuniversal** *iff for every metric space $\langle T, \nu \rangle$ of cardinality κ every isometry in $\mathscr{S}\mathscr{I}_\kappa(T,S)$ can be extended to an isometry in $\mathscr{I}(T,S)$.*

We begin our study of these spaces with a collection of alternate characterizations of this property. Of these, property 1 is the weakest in appearance, 2 is the one we shall generalize into what is called weak κ-superuniversality when we deal with singular cardinals in Section 4, 4 will have an interesting category theoretical interpretation which we shall mention in 5.4, and 6 is a version of 1 which is perhaps the most useful in actually proving that particular spaces are κ-superuniversal. Finally, we remember that to construct Urysohn's space, one first constructs a certain countable metric space and then looks at its completion. However, it follows immediately from the definition, that κ-superuniversal spaces are complete for every uncountable cardinal κ. (If $\mathscr{S}$ is any Cauchy sequence in such a space, then take the countable space consisting of said sequence plus a limit point and extend the identity isometry to obtain a limit point in the large space.) Thus it is not surprising to find that we can embed not only spaces of cardinality κ into κ-superuniversal spaces, but all spaces which contain dense subsets of cardinality κ. Unfortunately, this result is not quite as strong as it might seem due to the fact that if a metric space contains a dense subset of cardinality κ, then it can have cardinality at most $\kappa^{\aleph_0}$ which, in many cases, is just κ itself. However, it is still of interest, and we include it as part 5.

3.1. Theorem. *For any uncountable cardinal κ and any metric space $S = \langle S, \mu \rangle$ the following are equivalent:*

1. *If $\langle T, v \rangle$ is any metric space of cardinality less than κ and t is any member of T, then every isometry in $\mathscr{I}(T - \{t\}, S)$ can be extended to an isometry in $\mathscr{I}(T, S)$.*

2. *If $\langle T, v \rangle$ is any metric space of cardinality less than κ, then every member of $\mathscr{S}\mathscr{I}(T, S)$ can be extended to an isometry in $\mathscr{I}(T, S)$.*

3. *$\langle S, \mu \rangle$ is κ-superuniversal.*

4. *If $\langle T, v \rangle$ is any metric space of cardinality less than κ, $\langle U, \eta \rangle$ is any metric space of cardinality at most κ, $\boldsymbol{f}$ is any isometry in $\mathscr{I}(T, S)$, and $\boldsymbol{g}$ is any isometry in $\mathscr{I}(T, U)$, then there exists an isometry $\boldsymbol{h} \in \mathscr{I}(U, S)$ such that $\boldsymbol{h} \circ \boldsymbol{g} = \boldsymbol{f}$.*

5. *If $\langle U, v \rangle$ is any metric space which contains a dense subset V of cardinality κ, then every isometry in $\mathscr{S}\mathscr{I}_{\kappa}(U, S)$ can be extended to an isometry in $\mathscr{I}(U, S)$.*

6. *If f is any consistent function from some set $U \in \mathscr{P}_{\kappa}(S)$ into R^{+}, then there exists a point $s \in S$ such that*

$$\mu(s, u) = f(u) \qquad \text{for all } u \in U.$$

Proof. To obtain 2 from 1 or 3 from 2, apply the former an appropriate number of times using transfinite induction. Part 4 follows immediately from the fact that an isometry is an embedding, while 6 is an alternate statement of 1.

Finally, 5 can be obtained from 3 by first finding an extension $g \in \mathscr{I}(U \cup V, S)$ of f and then using the fact that S is complete. ■

We next consider the cardinality of κ-superuniversal spaces and begin with a lower bound.

3.2 THEOREM. *Every nonempty open subset of a κ-superuniversal metric space has cardinality at least $2^{\bar{\kappa}}$.*

PROOF. Let $\langle S, \mu \rangle$ be any κ-superuniversal space and let T be any nonempty open subset of S. Then for some positive real number r there exists a point $p \in T$ such that

$$\{t \in S : \mu(p, t) < r\} \subseteq T.$$

Now let λ be any cardinal less than κ. By the κ-superuniversality of $\langle S, \mu \rangle$, we may choose a set $U \subseteq S$ of cardinality λ which contains p and satisfies

$$\mu(u, v) = r \qquad \text{for all distinct } u, v \in U.$$

For each set $A \subseteq U$, let f^A be the function from U into R^+ defined by

$$f^A(u) = \begin{cases} r/2 & u \in A \\ 3r/4 & u \notin A. \end{cases}$$

Each function f^A is consistent so by 3.1.6 we have for each $A \subseteq U$ a point $p^A \in S$ satisfying

$$\mu(p^A, u) = \begin{cases} r/2 & u \in A \\ 3r/4 & u \notin A \end{cases} \qquad \text{for every } u \in U.$$

But clearly, each $p^A \in T$, and we have

$$p^A = p^B \leftrightarrow A = B,$$

so $\{p^A : A \subseteq U\}$ is a subset of T of cardinality 2^λ. Hence we have shown that

$$|T| \geqq 2^\lambda \qquad \text{for every } \lambda < \kappa,$$

and $|T|$ must therefore be at least $2^{\bar{\kappa}}$. ■

Thus every κ-superuniversal space has at least $2^{\bar{\kappa}}$ points. This will also follow from 3.12. For uncountable regular cardinals κ, this will be best possible because the $H_\kappa^{\prec}$ are, as we will note next, κ-superuniversal. For singular cardinals we do not know. The most that can be said at the moment is that if κ is singular and $2^{\bar{\kappa}} = \kappa$, then there do not exist κ-superuniversal spaces of cardinality $2^{\bar{\kappa}}$. We prove this in 4.6.

3.3 Theorem. *If κ is any uncountable regular cardinal and $\prec$ is any admissible well ordering of $\mathscr{F}_\kappa$, then $H_\kappa^\prec$ is κ-superuniversal.*

Proof. Immediate from 2.3 and 3.1.6. ■

3.4 Corollary. *If κ is either an uncountable regular cardinal or a singular cardinal for which some cardinal λ satisfies $2^\lambda = 2^{\tilde{\kappa}}$, then:*

1. *The smallest κ-superuniversal metric space has cardinality $2^{\tilde{\kappa}}$.*

2. *There exists a κ-superuniversal metric space of cardinality κ iff there exists a cardinal γ such that $\kappa = \gamma^+ = 2^\gamma$ or κ is inaccessible and every cardinal $\gamma < \kappa$ satisfies $2^\gamma \leqq \kappa$.*

Proof. For regular cardinals these properties follow directly from 3.2, 3.3, and the definition of $2^{\tilde{\kappa}}$. If, however, κ is singular and some cardinal $\lambda < \kappa$ satisfies $2^\lambda = 2^{\tilde{\kappa}}$, then by a theorem of the author [7] and Bukovský [1] we have $2^{\tilde{\kappa}} = 2^\kappa$. But $2^\kappa = 2^{\widetilde{\kappa^+}}$, so it is sufficient to look at spaces of the form $H_{\kappa^+}^\prec$. ■

Using the above, we can partially answer the question as to when there exist κ-superuniversal metric spaces of cardinality κ. The best we can hope for, of course, is a collection of consistency results and even here we cannot obtain complete results because we do not know enough about powers of singular cardinals. In particular, we do not know if it is consistent to assume that for no cardinals κ do there exist κ-superuniversal spaces of cardinality κ; the best we can do in this direction is 3.5.4 below.

3.5 Theorem.

1. *The following statements are equivalent:*

   a. *For every regular uncountable cardinal κ, there exists a κ-superuniversal metric space of cardinality κ.*

   b. *For every successor cardinal κ, there exists a κ-superuniversal metric space of cardinality κ.*

   c. *(Generalized Continuum Hypothesis) Every infinite cardinal κ satisfies $2^{\tilde{\kappa}} = \kappa$.*

2. *It is consistent with the axioms of Zermelo-Fraenkel set theory with choice (ZFC) that, for every regular uncountable cardinal κ, there exist κ-superuniversal metric spaces of cardinality κ.*

3. *It is consistent with ZFC that, for some but not all regular cardinals κ, there exist κ-superuniversal metric spaces of cardinality κ.*

4. *It is consistent with ZFC that every regular cardinal κ, for which there*

exists a κ-superuniversal metric space of cardinality κ, be the successor of a singular cardinal.

5. *The existence of a strongly compact cardinal implies the existence of arbitrarily large regular cardinals κ for which there exist κ-superuniversal metric spaces of cardinality κ.*

PROOF. Part 1 follows from 3.4, part 2 follows from the known [6] consistency of the generalized continuum hypothesis, while parts 3 and 4 follow from 3.4 and Easton's [2; 3] consistency results on the function $2^{\bar{\kappa}}$. Part 5 follows from a result recently announced by Solovay and Kunen. A definition of and information about strongly compact cardinals can be found in [13]. ■

We next consider the question of uniqueness. We show that for uncountable regular cardinals κ there exists a unique (up to isometry) κ-superuniversal space iff $2^{\bar{\kappa}} = \kappa$. We begin with:

3.6 THEOREM *If κ is any uncountable regular cardinal satisfying $2^{\bar{\kappa}} = \kappa$, then all κ-superuniversal metric spaces of cardinality κ are isometric.*

PROOF. Suppose $\mathbf{S} = \langle S, \mu \rangle$ and $\mathbf{T} = \langle T, v \rangle$ are κ-superuniversal metric spaces of cardinality κ. We well order both S and T with order type κ thus obtaining $S = \{s_\alpha : \alpha < \kappa\}$ and $T = \{t_\alpha : \alpha < \kappa\}$, and using these well orderings we construct inductively certain sequences $\{f_\alpha : \alpha < \kappa\} \subset \mathcal{S}\mathcal{I}_\kappa(S, T)$ and $\{g_\alpha : \alpha < \kappa\} \subset \mathcal{S}\mathcal{I}_\kappa(T,S)$. Using 3.1.1 we choose a function Φ such that

$$\Phi(f,\alpha) \text{ extends } f \text{ for all } f \in [\mathcal{S}\mathcal{I}_\kappa(S,T) \cup \mathcal{S}\mathcal{I}_\kappa(T,S)],$$

$$\Phi(f,\alpha) \in \mathcal{S}\mathcal{I}_\kappa(S,T) \text{ and } s_\alpha \in \mathrm{dm}(\Phi(f,\alpha)) \text{ for all } f \in \mathcal{S}\mathcal{I}_\kappa(S,T),$$

$$\Phi(f,\alpha) \in \mathcal{S}\mathcal{I}_\kappa(T,S) \text{ and } t_\alpha \in \mathrm{dm}(\Phi(f,\alpha)) \text{ for all } f \in \mathcal{S}\mathcal{I}_\kappa(T,S).$$

Using this function, we inductively define our sequences by setting

$$f_0 = \{\langle s_0, t_0 \rangle\} \text{ and } g_0 = \{\langle t_0, s_0 \rangle\},$$

$$f_{\alpha+1} = \Phi(g_\alpha^{-1}, \alpha+1) \text{ and } g_{\alpha+1} = \Phi(f_{\alpha+1}^{-1}, \alpha+1) \text{ for all } \alpha < \kappa,$$

$$f_\lambda = \bigcup_{\alpha<\lambda} f_\alpha \text{ and } g_\lambda = \bigcup_{\alpha<\lambda} g_\alpha \text{ for all limit ordinals } \lambda < \kappa.$$

It is easily seen that these are internally consistent and that

$$f = \bigcup_{\alpha<\kappa} f_\alpha \in \mathcal{I}(S,T).$$ ■

Thus, in what follows, for regular uncountable cardinals satisfying $2^{\bar{\kappa}} = \kappa$ we use H_κ to denote the unique (up to isometry) κ-superuniversal metric space of cardinality κ.

To prove that when $2^{\bar{\kappa}} \neq \kappa$, there exist κ-superuniversal spaces of cardinality $2^{\bar{\kappa}}$ other than the $H_\kappa^{\prec}$, we define a certain kind of subspace called a κ-isolated unitary space and prove that while every such subspace of $H_\kappa^{\prec}$ has cardinality at most κ, it is possible to modify the construction of $H_\kappa^{\prec}$ in such a way as to obtain a κ-superuniversal space of cardinality $2^{\bar{\kappa}}$ which contains a κ-isolated unitary subspace which is of cardinality $2^{\bar{\kappa}}$. We need some definitions and lemmas.

First define a metric space $\langle S, \mu \rangle$ to be a **unitary space** iff

$$\mu(s,t) = 1 \qquad \text{for all distinct } s,t \in S.$$

Then if $\langle S, \mu \rangle$ is any metric space, κ is any cardinal, T is any unitary subspace of S, and p is any point in $S - T$, define p to be **strongly κ-isolated** from T iff

$$\left| \{ t \in T : \mu(p,t) < 3/2 \} \right| < \kappa;$$

to be **weakly κ-isolated** from T with respect to some point $u \in T$ iff

$$\mu(p,u) < \tfrac{1}{2} \text{ and}$$

$$\left| \{ t \in T : \mu(p,t) \neq 1 + \mu(p,u) \} \right| < \kappa;$$

and to be **κ-isolated** from T iff it is either strongly or weakly (with respect to some point $u \in T$) κ-isolated from T. Finally, define T itself to be **κ-isolated** in S iff every point in $S - T$ is κ-isolated from T.

The notion of κ-isolation seems to be necessary because if p is any point in an $H_\kappa^{\prec}$, then there will grow around p a unitary space each of whose points is exactly $\frac{1}{2}$ from p and it is easily seen from the construction of $H_\kappa^{\prec}$ that this space will eventually contain $2^{\bar{\kappa}}$ points. The next theorem, which contains the crucial part of our argument, in effect tells us that κ-isolated unitary spaces do not grow too large in the $H_\kappa^{\prec}$ but, in fact, have cardinality at most κ.

3.7 THEOREM. *Let $\langle S, \mu \rangle$ be any metric space, let κ be any regular cardinal, let T be any κ-isolated subspace of S, and let f be any consistent nonsuperfluous function from some set $D \in \mathscr{P}_\kappa(S)$ into R^+. Then if we add a new point p to S and extend μ by setting*

$$\mu(p,d) = f(d) \qquad \text{for } d \in D, \text{ and}$$

$$\mu(p,s) = \inf(\{ f(d) + \mu(d,s) : d \in D \}) \text{ otherwise,}$$

the point p will be κ-isolated from T.

PROOF. For each element $d \in D$, we define the pathological set P_d of d by

$$P_d = \{d\} \qquad \text{if } d \in T,$$

$$P_d = \{t \in T : \mu(d,t) < 3/2\} \qquad \text{if } d \text{ is strongly } \kappa\text{-isolated from } T,$$

$$P_d = \{t \in T : \mu(d,t) \neq 1 + \mu(d,u)\} \qquad \text{if } d \text{ is weakly } \kappa\text{-isolated from } T \text{ with respect to } u,$$

and we define the pathological set P_D of D by

$$P_D = \bigcup_{d \in D} P_d .$$

We note that since T is κ-isolated, each P_d has cardinality less than κ. But κ is regular, and D also has cardinality less than κ, so we may conclude that

$$|P_D| < \kappa.$$

We now consider two separate cases.

Case 1. Suppose we have

$$f(d) + \mu(d,t) \geqq \tfrac{1}{2} \quad \text{for all } d \in D \text{ and } t \in T.$$

We show that, in this case, p is strongly κ-isolated from T. In particular, for any fixed $t_0 \in T - P_D$, we have

$$\mu(p, t_0) \geqq 3/2.$$

To prove this, it is sufficient from the definition of μ to prove that

$$f(d) + \mu(d, t_0) \geqq \tfrac{3}{2} \quad \text{for all } d \in D.$$

But $t_0 \notin P_d \subseteq P_D$ so either $\mu(d,t_0) \geqq \frac{3}{2}$ in which case we are done, or for some $u \in T$ we have

$$\mu(d,u) < \tfrac{1}{2} \text{ and } \mu(d,t_0) = 1 + \mu(d,u).$$

However, our assumption in case 1 is that for no $u \in T$ do we have $f(d) + \mu(d,u) < \frac{1}{2}$. Thus we see that

$$f(d) + \mu(d,t_0) = f(d) + \mu(d,u) + 1 \geqq (\tfrac{1}{2}) + 1 = \tfrac{3}{2}.$$

Case 2. For some $d \in D$ and some $u \in T$, we have

$$f(d) + \mu(d,u) < \tfrac{1}{2}.$$

We note that although d is not determined by the above inequality, it follows from the consistency of f and the fact that T is unitary that u is determined. Thus we may think of u as fixed. We note from the definition of μ that

$$u(p,u) < \tfrac{1}{2},$$

and we shall show that p is weakly κ-isolated from T with respect to u. Again, in particular, we shall show that for any fixed $t_0 \in T - P_D$ we have

$$\mu(p,t_0) = 1 + \mu(p,u).$$

Since the triangle inequality gives us

$$\mu(p,t_0) \leqq \mu(t_0,\mu) + \mu(p,u) = 1 + \mu(p,u),$$

we must prove

$$\inf(\{f(d) + \mu(d,t_0)\colon d \in D\}) \geqq 1 + \inf(\{f(d) + \mu(d,u)\colon d \in D\}).$$

Equivalently, for every $d \in D$ we must exhibit a point $e \in D$ such that

$$1 + f(e) + \mu(e,u) \leqq f(d) + \mu(d,t_0).$$

Thus choose any fixed $d \in D$. Since $t_0 \notin P_d \subseteq P_D$ we again have either $\mu(d,t_0) \geqq \tfrac{3}{2}$ in which case we are done because of the hypothesis defining case 2 or we have some $v \in T$ such that

$$\mu(d,v) < \tfrac{1}{2} \text{ and } \mu(d,t_0) = 1 + \mu(d,v).$$

If $v = u$ we are again done because our desired inequality becomes

$$1 + f(e) + \mu(e,u) \leqq f(d) + \mu(d,t_0) = f(d) + 1 + \mu(d,u),$$

and we may simply set $e = d$. On the other hand, suppose that $v \neq u$. Then from the fact that T is unitary and from the definition of μ we observe that

$$\mu(p,u) < \tfrac{1}{2} \text{ and } \mu(u,v) = 1,$$

so by the triangle inequality and another appeal to the definition of μ we have

$$\tfrac{1}{2} < \mu(p,v) \leqq f(d) + \mu(d,v).$$

Hence for any $e \in D$ satisfying

$$f(e) + \mu(e,u) < \tfrac{1}{2},$$

we have

$$1 + f(e) + \mu(e,u) < 1 + \tfrac{1}{2} \leqq 1 + f(d) + \mu(d,u) = f(d) + \mu(d,t_0). \qquad \blacksquare$$

Using this, we can prove:

3.8 Theorem. *If κ is any uncountable regular cardinal, $\prec$ is any admissible well ordering of $\mathscr{F}_\kappa$, and T is any κ-isolated unitary subspace of $H_\kappa^{\prec}$, then T has cardinality at most κ.*

PROOF. Suppose not. Let α be any member of T such that

$$|T \cap \alpha| = \kappa.$$

Then $T \cap \alpha$ is a κ-isolated unitary subspace of $\langle \alpha, \mu \restriction a \rangle$ of cardinality κ. But from 3.7 and the construction of $H_\kappa^{\prec}$, we see that the point α itself must be κ-isolated from $T \cap \alpha$. Thus we cannot have

$$|\{\beta \in (T \cap \alpha) : \mu(\beta, \alpha) = 1\}| = \kappa. \qquad \blacksquare$$

3.9 THEOREM. *If κ is any uncountable regular cardinal, then there exists a κ-superuniversal metric space S_κ of cardinality $2^{\bar{\kappa}}$ which contains a κ-isolated unitary subspace also of cardinality $2^{\bar{\kappa}}$.*

PROOF. Our construction of S_κ will proceed in almost the same manner as that of $H_\kappa^{\prec}$. Thus let $\prec$ be an admissible well ordering of $\mathscr{F}_\kappa$, again use $H_\kappa = 2^{\bar{\kappa}}$ as our ground set, and inductively define our metric ν point by point. In fact for any ordinal γ of the form $\alpha + 1$, we extend ν to γ *exactly* as before. If, however, γ is a limit ordinal, we use as f^γ not some least member of $\mathscr{F}_\kappa$, but the function defined by

$$f^\gamma(\delta) = 1 \qquad \text{for every limit ordinal } \delta < \gamma.$$

Since each new f^γ is both consistent and nonsuperfluous, the construction presents no new problems, and the proof that S_κ is κ-superuniversal is as in 3.3.

It is also easily seen that the set

$$T = \{\gamma < 2^{\bar{\kappa}} : \gamma \text{ is a limit ordinal}\}$$

is a κ-isolated unitary subset of S_κ which has cardinality $2^{\bar{\kappa}}$. For, suppose not. Then there must be some least ordinal $\alpha \in 2^{\bar{\kappa}} - T$ such that α is not κ-isolated from T. Thus $T \cap \alpha$ must be κ-isolated in $\langle \alpha, \nu \restriction \alpha \rangle$. Hence by 3.7 and the construction of ν, α itself must be κ-isolated from $T \cap \alpha$. But it also follows from the definition of ν that for each $t \in T - \alpha$ we have

$$\nu(t, \alpha) = \inf(\{1 + \nu(d, \alpha) : d \in T \cap \alpha\}),$$

and thus α is, in fact, κ-isolated from all of T. $\blacksquare$

Finally, by combining 3.8 and 3.9, we see that not only do we have at least two nonisometric κ-superuniversal metric spaces of cardinality $2^{\bar{\kappa}}$ wherever κ is an uncountable regular cardinal such that $\kappa < 2^{\bar{\kappa}}$, but:

3.10 COROLLARY. *For every uncountable regular cardinal κ such that $\kappa < 2^{\bar{\kappa}}$*

there exists a κ-superuniversal metric space S_κ of cardinality $2^{\bar{\kappa}}$ which is not isometric to any $H_\kappa^{<}$. ∎

We conclude this section with a study of κ-superuniversal subspaces of κ-superuniversal spaces. We shall prove that every such space contains a family of $2^{\bar{\kappa}}$ disjoint such subspaces, and we shall construct such spaces of cardinality $2^{\bar{\kappa}}$ which can be decomposed into a disjoint union of $2^{\bar{\kappa}}$ such subspaces. We begin with:

3.11 THEOREM. *Let $\langle S,\mu\rangle$ be any κ-superuniversal metric space, let T be any subset of S of cardinality less than κ, and let t be any fixed member of T. Then the space generated by the set*

$$U = \{s \in S : u \in T \rightarrow \mu(s,u) = \mu(s,t) + \mu(t,u)\}$$

is itself κ-superuniversal.

PROOF. Use 3.1.6 as the characterization of κ-superuniversality. Thus let f be any consistent function from some member $V \in \mathscr{P}_\kappa(U)$ into R^+. If f is superfluous at the fixed point t, then it follows from 2.1.2 that

$$\mu(t,v) = f(v) \quad \text{for every } v \in V\,,$$

and, since it follows from our definition that $t \in U$, we are done. If f is not superfluous at t, we may assume that t is a member of its domain, for if not we can set

$$f(t) = \inf(\{f(v) + \mu(v,t) : v \in V\})\,,$$

and f will remain consistent. But now we can consistently extend f to all of T by setting

$$f(s) = f(t) + \mu(s,t) \quad \text{for all } s \in T.$$

The expanded function f is a consistent function from the set $V \cup T \in \mathscr{P}_\kappa(S)$ in R^+, and thus from the κ-superuniversality of $\langle S,\mu\rangle$, there must exist a point $p \in S$ such that

$$u(p,s) = f(s) \quad \text{for all } s \in V \cup T.$$

Then, in particular, we have

$$\mu(p,s) = f(t) + \mu(s,t) = \mu(p,t) + \mu(t,s) \qquad \text{for all } s \in T,$$

so $p \in U$ and we have satisfied 3.1.6. ∎

This can be used to give us:

3.12 COROLLARY. *If $\langle S,\mu\rangle$ is any κ-superuniversal metric space, then there exists a family of $2^{\bar{\kappa}}$ disjoint κ-superuniversal subsets of S.*

PROOF. We first prove that we can find a family of κ such spaces. Let $U = \{u_\alpha : \alpha < \kappa\}$ be a unitary subspace of S, and for each $0 < \alpha < \kappa$ define

$$T^\alpha = \{s \in S : \beta < \alpha \to \mu(s,u_\beta) = 1 + \mu(s,u_\alpha)\}.$$

We see that if we regard $\{u_\beta : \beta \leqq \alpha\}$ as T and u_α as t, we can apply 3.11 to show that each T^α is κ-superuniversal. Also, for any $0 < \alpha < \beta < \kappa$, we have

$$\mu(t,u_0) = 1 + \mu(t,u_\beta) = \mu(t,u_\alpha) \qquad \text{for all } t \in T^\beta,$$

$$\mu(t,u_0) = 1 + \mu(t,u_\alpha) \qquad \text{for all } t \in T^\alpha,$$

so

$$\alpha \neq \beta \to T^\alpha \cap T^\beta = \varnothing \qquad \text{for all } 0 < \alpha < \beta < \kappa.$$

Thus if $2^{\bar{\kappa}} = \kappa$, we are done. Otherwise, we also need to know that for any cardinal $\lambda < \kappa$ we can find a family of 2^λ disjoint κ-superuniversal subspaces. To show this, choose some unitary subspace $V \subseteq S$ of cardinality λ and for each set $A \subseteq V$ choose a point t^A such that

$$\mu(t^A,v) = \begin{cases} \frac{1}{2} & v \in A \\ \frac{3}{4} & v \in V - A. \end{cases}$$

Now, as before, let

$$T^A = \{s \in S : v \in V \to \mu(s,v) = \mu(s,t^A) + \mu(t^A,v)\} \qquad \text{for all } A \subseteq V.$$

Again, by 3.11, each T^A is κ-superuniversal, and by an argument similar to the above, we have

$$A \neq B \to U^A \cap U^B = \varnothing \qquad \text{for all } A,B \subseteq V.$$

Finally, let $\{F_\alpha : \alpha < \kappa\}$ be a family of disjoint κ-superuniversal subsets of S, and for each $\alpha < \kappa$, let $\{F_\alpha^\beta : \beta < 2^\alpha\}$ be a family of disjoint κ-superuniversal subsets of F_α. Then the family

$$\{F_\alpha^\beta : \alpha < \kappa \text{ and } \beta < 2^\alpha\}$$

is a family of $2^{\bar{\kappa}}$ disjoint κ-superuniversal subsets of S. ■

We should note that while all of the κ-superuniversal subsets of κ-superuniversal spaces that we have constructed are nowhere dense, this is not a property of all such subsets. In fact, using essentially the same techniques as in the proof of 3.11, we can show:

3.13 THEOREM *If $\langle S,\mu\rangle$ is any κ-superuniversal metric space, p is any point in S and r is any positive real number, then the subset*

$$T = \{s \in S: \mu(s,p) \geqq r\}$$

is κ-superuniversal. ∎

On the other hand, every κ-superuniversal subspace is closed, and since even $\aleph_1$-superuniversal spaces are arcwise connected (by 3.1.5), we cannot decompose a κ-superuniversal space into a *finite* union of disjoint κ-superuniversal spaces. In general, we do not know about infinite such decompositions; the best we can do is:

3.14 THEOREM. *For every uncountable regular cardinal κ there exists a κ-superuniversal space which can be decomposed into a disjoint union of $2^{\bar{\kappa}}$ isometric κ-superuniversal subspaces.*

PROOF. Let κ be any uncountable regular cardinal. Our construction will consist of pasting together $2^{\bar{\kappa}}$ copies of some $H_\kappa^{\prec}$. Thus let $\prec$ be an admissible well ordering of $\mathscr{F}_\kappa$, and choose an indexed set

$$S = \{s_\alpha^\beta : \alpha, \beta < 2^{\bar{\kappa}}\}$$

which will be of cardinality $(2^{\bar{\kappa}})^2 = 2^{\bar{\kappa}}$. Then set

$$\mu(s_\alpha^\beta, s_\gamma^\beta) = \mu^{\prec}(\alpha,\gamma) \qquad \text{for all } \alpha,\beta,\gamma < 2^{\bar{\kappa}}$$

where $\mu^{\prec}$ is just the metric on $H_\kappa^{\prec}$. We let $\mathscr{G}$ be the set of functions from $\mathscr{P}_\kappa(S)$ into R^+, and we choose a well ordering $\prec$ of $\mathscr{G}$ under which $\mathscr{G}$ is order isomorphic to $2^{\bar{\kappa}}$($\mathscr{G}$ has cardinality $2^{\bar{\kappa}}$by 2.2).

We shall define the metric μ between points in the different copies of $H_\kappa^{\prec}$ by induction using $\mathscr{G}$. Thus assume we have already defined $\mu(s_\alpha^\beta, s_\gamma^\delta)$ for all β, $\delta < \sigma$ and that we wish to define $\mu(s_\alpha^\beta, s_\gamma^\sigma)$ for $\beta < \sigma$. Let g be the least member of $\mathscr{G}$ under $\prec$ satisfying

$$s_\alpha^\beta \in \mathrm{dm}(g) \rightarrow \beta < \sigma$$

which has not yet been used and is consistent but not superfluous and set

$$\mu(s_0^\sigma, s_\alpha^\beta) = \inf(\{g(s_\gamma^\delta) + \mu(s_\gamma^\delta, s_\alpha^\beta) : s_\gamma^\delta \in \mathrm{dm}(g)\}) \quad \text{for all } \beta < \sigma,$$

$$\mu(s_\gamma^\sigma, s_\alpha^\beta) = \mu(s_\gamma^\sigma, s_0^\sigma) + \mu(s_0^\sigma, s_\alpha^\beta) \qquad \text{for all } \beta < \sigma.$$

It is easily seen that the resulting metric has all the desired properties. ∎

3.15 COROLLARY. *If κ is any regular cardinal satisfying $2^{\bar{\kappa}}=\kappa$, then $\boldsymbol{H}_\kappa$ can be decomposed into a union of κ disjoint κ-superuniversal subspaces each isometric to the original space.* ∎

4. Weak superuniversality and singular cardinals

As stated earlier, when dealing with singular cardinals, we need certain boundedness conditions. Therefore define an **ordered metric space** to be a structure $\langle S, \mu, \prec \rangle$ such that $\langle S, \mu \rangle$ is a metric space and $\langle S, \prec \rangle$ is a well-ordered structure order isomorphic to $|S|$. Since we shall frequently be concerned with isometries with bounded ranges, define, for any metric space $\langle T, \nu \rangle$ and any ordered metric space $\langle S, \mu, \prec \rangle$, the sets $\mathscr{B}\mathscr{I}(T,S)$, $\mathscr{B}\mathscr{S}\mathscr{I}(T,S)$, $\mathscr{B}\mathscr{S}\mathscr{I}_\kappa(T,S)$, and $\mathscr{B}\mathscr{S}\mathscr{I}_\kappa^+(T,S)$ by

$$\mathscr{B}(\mathscr{S})\mathscr{I}^{(+)}_{(\kappa)} = \{f \in (\mathscr{S})\mathscr{I}^{(+)}_{(\kappa)}(T,S) \colon \mathrm{rn}(f) \in \mathscr{B}(S)\}.$$

Then, following 3.1.2, define an ordered metric space $\boldsymbol{S}$ to be **weakly κ-superuniversal** iff, for each metric space $\boldsymbol{T}$ of cardinality less than κ, every isometry $f \in \mathscr{B}\mathscr{S}\mathscr{I}(T,S)$ can be extended to an isometry $g \in \mathscr{B}\mathscr{I}(T,S)$.

It should be noted that we do not require $\boldsymbol{T}$ to be ordered nor do we require that f and or g be order preserving. Such conditions would be impossible to fulfill because Cauchy sequences have unique limits in any given space whereas the positions of these limits in an ordering are not at all unique.

We note that there do not seem to be any general implications between the notions of weak κ-superuniversality, and either κ-superuniversality or even universality for all metric spaces of cardinality κ. The best we can do is:

4.1 THEOREM. *For any metric space $\langle S, \mu \rangle$:*

1. *If $\mathrm{cf}(|S|) \geqq \kappa$, then $\boldsymbol{S}$ is κ-superuniversal iff for every well ordering $\prec$ of S such that $\langle S, \prec \rangle$ has order type $|S|$, the ordered metric space $\langle S, \mu, \prec \rangle$ is weakly κ-superuniversal.*
2. *If $\mathrm{cf}(|S|) \geqq \mathrm{cf}(\kappa)$ and $\langle S, \mu \rangle$ is weakly κ-superuniversal, then:*
   a. *If $\langle T, \nu \rangle$ is any metric space of cardinality κ, then every member of $\mathscr{B}\mathscr{S}\mathscr{I}_\kappa(T,S)$ can be extended to an isometry in $\mathscr{I}(T,S)$.*

   b. *If $\langle T, \nu \rangle$ is any metric space of cardinality less than κ, $\langle U, \eta \rangle$ is any metric space of cardinality at most κ, f is any isometry in $\mathscr{B}\mathscr{I}(T,S)$, and g is any isometry in $\mathscr{I}(T,U)$, then there exists an isometry $h \in \mathscr{I}(U,S)$ such that $h \circ g = f$.*

   c. *If $|S|$ has cofinality greater than $\aleph_0$ (in particular, if $|S| = 2^{\bar{\kappa}}$ and $\mathrm{cf}(\kappa) > \aleph_0$), then:*

1. *$\langle S,\mu\rangle$ is complete.*

2. *If $\langle U,v\rangle$ is any metric space which contains a dense subset of cardinality κ, then every isometry in $\mathscr{BSI}_\kappa(U,S)$ can be extended to an isometry in $\mathscr{I}(U,S)$.*

PROOF. Part 1 follows immediately from the appropriate definitions. To construct the desired isometry in 2.a, we divide T into cf(κ) pieces each of cardinality less than κ and then construct an increasing sequence of bounded isometries piece by piece. Parts b and c follow from a. ∎

Using essentially the same proof as in 3.2, we have:

4.2 THEOREM. *Every nonempty open subset of a weakly κ-superuniversal ordered metric space has cardinality at least $2^{\bar{\kappa}}$.* ∎

We also have:

4.3 THEOREM. *If $\langle S,\mu,\prec\rangle$ and $\langle T,v,\prec\rangle$ are any two weakly κ-superuniversal metric ordered spaces of cardinality κ, then $\langle S,\mu\rangle$ and $\langle T,v\rangle$ are isometric.*

PROOF. The construction of the isometry is similar to the construction of the analogous isometry in the proof of 3.6 except that it is not extended one element at a time but rather along an unbounded sequence of cardinality $\lambda = \mathrm{cf}(\kappa)$. Thus let $\mathscr{S} = \{s_\alpha : \alpha < \lambda\}$ and $\mathscr{T} = \{t_\alpha : \alpha < \lambda\}$ be strictly increasing unbounded sequences of elements from S and T respectively. Almost as in 3.6, let $\mathscr{F}$ be the family of isometries from members of $\mathscr{B}_\kappa(S)$ into members of $\mathscr{B}_\kappa(T)$, let $\mathscr{G}$ be the family of isometries from members of $\mathscr{B}_\kappa(T)$ into members of $\mathscr{B}_\kappa(S)$, and again choose Φ such that for all $\alpha < \lambda$:

$$\Phi(f,\alpha) \text{ extends } f \qquad \text{for all } f \in \mathscr{F} \cup \mathscr{G},$$

$$\Phi(f,\alpha) \in \mathscr{F} \text{ and } \{s \in S : s \prec s_\alpha\} \in \mathrm{dm}(\Phi(f,\alpha)) \qquad \text{for all } f \in \mathscr{F},$$

$$\Phi(g,\alpha) \in \mathscr{G} \text{ and } \{t \in T : t \prec t_\alpha\} \in \mathrm{dm}(\Phi(g,\alpha)) \qquad \text{for all } g \in \mathscr{G}.$$

Next, define two sequences $\{f_\alpha \in \mathscr{F} : \alpha < \lambda\}$ and $\{g_\alpha \in \mathscr{G} : \alpha < \lambda\}$ inductively by setting:

$$f_{\alpha+1} = \Phi(g_\alpha^{-1}, \alpha+1) \text{ and } g_{\alpha+1} = \Phi(f_{\alpha+1}, \alpha+1) \qquad \text{for all } \alpha < \lambda,$$

$$f_\gamma = \bigcup_{\alpha<\gamma} f_\alpha \text{ and } g_\gamma = \bigcup_{\alpha<\gamma} f_\gamma \text{ otherwise.}$$

It is easily seen that each $f_\beta \in \mathscr{F}$ and each $g_\beta \in \mathscr{G}$. For $\beta = \alpha + 1$ this follows from the definition of Φ, and for β a limit ordinal it follows from the fact that since κ has cofinality λ, a union of fewer than λ bounded subsets of S or T

remains bounded. Thus the construction is well defined for all $\alpha < \lambda$, and $f = \bigcup_{\alpha<\lambda} f_\alpha$ is an isometry from $\langle S, \mu \rangle$ onto $\langle T, \nu \rangle$. ∎

We are now ready to consider the weak superuniversality of the spaces $H_\kappa^\prec$. We have already shown (3.3) that, for κ regular and $\prec$ admissible, these spaces are κ-superuniversal, so by 4.1.3 they are, under any appropriate well ordering and in particular under $<$, weakly κ-superuniversal. Similarly, if for some $\lambda < \kappa$ we have $2^\lambda = 2^{\bar{\kappa}}$ and κ is singular, then, as noted in the proof of 3.4, the spaces $H_\kappa^\prec$ and $H_{\kappa^+}^\prec$ have the same cardinality and we need not concern ourselves with the former. Thus we define a singular cardinal κ to be **strongly singular** iff

$$2^\lambda < 2^{\bar{\kappa}} \text{ for all } \lambda < \kappa.$$

We note that a singular cardinal is strongly singular iff

$$\mathrm{cf}(2^{\bar{\kappa}}) = \mathrm{cf}(\kappa).$$

We begin our study of strongly singular cardinals with a return to the construction of the $H_\kappa^\prec$. As we have mentioned, we must put additional restrictions on the ordering $\prec$. We first define the **height** of a function $f \in \mathscr{F}_\kappa$ by

$$\mathrm{ht}(f) = \max(\mathrm{bn}(f), 2^{dm(f)}).$$

Then for κ strongly singular and having cofinality λ, and $\mathscr{S} = \{\gamma_\beta : \beta < \lambda\}$ an unbounded sequence of cardinals in κ, we define an admissible well ordering $\prec$ of $\mathscr{F}_\kappa$ (which becomes $\{f_\alpha : \alpha < \kappa\}$) to be **strongly $\mathscr{S}$-admissible** iff we have

$$\mathrm{ht}(f_\alpha) \leqq 2^{\lambda_\beta} \to \alpha < 2^{\lambda_\beta} \qquad \text{for all } \alpha < \kappa \text{ and } \beta < \lambda.$$

Finally, we define an admissible well ordering of $\mathscr{F}_\kappa$ to be **strongly admissible** iff it is strongly $\mathscr{S}$-admissible for some appropriate sequence $\mathscr{S}$.

4.4 LEMMA. *If κ is any strongly singular cardinal with cofinality λ, then:*

1. *If $\mathscr{S} = \{\gamma_\beta : \beta < \lambda\}$ is any unbounded sequence of infinite cardinals in κ satisfying*

$$\sum_{\alpha<\beta} 2^{\gamma_\alpha} < 2^{\gamma_\beta} \qquad \textit{for all } \beta < \lambda,$$

then there exists a strongly $\mathscr{S}$-admissible well ordering of $\mathscr{F}_\kappa$.

2. *If $\mathscr{S} = \{\gamma_\beta : \beta < \lambda\}$ is any unbounded sequence of infinite cardinals in κ, $\prec$ is any strongly $\mathscr{S}$-admissible well ordering of $\mathscr{F}_\kappa$, $\langle T, \nu \rangle$ is any metric space of cardinality $\sigma < \kappa$, t is any member of T, f is any isometry from $T - \{t\}$ onto some set $S \in \mathscr{B}_\kappa(H_\kappa)$, and β is any ordinal such that*

$$2^{\sigma} < 2^{\lambda_\beta} \text{ and } S \subseteq 2^{\lambda_\beta},$$

then f can be extended to an isometry g from T into H_κ such that $g(t) < 2^{\gamma_\beta}$.

PROOF. Part 1 follows from the fact that for any $\mathscr{B} < \lambda$ we have

$$\left|\{f \in \mathscr{F}_\kappa : \operatorname{ht}(f) \leqq 2^{\gamma_\beta}\}\right| = (2^{\gamma_\beta})^{\gamma_\beta} \cdot (2^{\aleph_0})^{\gamma_\beta} = 2^{\gamma_\beta}.$$

The proof of part 2 will follow almost immediately from the definition of strong admissibility and is, in fact, the motivation behind this definition. We begin by letting f be the function defined by

$$f(f(u)) = v(u,t) \qquad \text{for all } u \in T - \{t\}.$$

From 2.1.1 it follows that f is consistent. Because

$$\operatorname{dm}(f) = \operatorname{rn}(f) = S \in \mathscr{P}_\sigma^+(2^{\gamma_\beta}) \subset \mathscr{B}_\kappa(H_\kappa),$$

we have $f \in \mathscr{F}_\kappa$, and because our hypotheses imply

$$2^{\operatorname{dm}(f)} = 2^{\sigma} \leqq 2^{\gamma_\beta},$$

we also see that

$$\operatorname{ht}(f) \leqq 2^{\gamma_\beta}.$$

Now suppose that under the well ordering $\prec$ of $\mathscr{F}_\kappa$, we have $f = f_\delta$. Then by the strong $\mathscr{S}$-admissibility of $\prec$, we have $\delta < 2^{\gamma_\beta}$.
Finally, by applying König's lemma (which implies that

$$\operatorname{cf}(2^{\eta}) > \eta \text{ for every infinite cardinal } \eta)$$

to S which has cardinality $\sigma < \gamma_\beta$, we obtain

$$\operatorname{bn}(f) < 2^{\lambda_\beta}.$$

Thus we can apply 2.3 to obtain a point $\alpha < 2^{\gamma_\beta}$ satisfying

$$\mu^{\prec}(\alpha, f(u)) = f(f(u)) = v(t,u) \qquad \text{for all } u \in T - \{t\},$$

and can set $g(t) = \alpha$. ∎

4.5 THEOREM. *If κ is any strongly singular cardinal with cofinality λ, then there exist strongly admissible well orderings of $\mathscr{F}_\kappa$.*

Moreover, if $\prec$ is any such strongly admissible well ordering then:

1. *$\langle H_\kappa, \mu^{\prec}, < \rangle$ is weakly κ-superuniversal.*
2. *$\langle H_\kappa, \mu^{\prec}, < \rangle$ is λ-superuniversal.*
3. *$\langle H_\kappa, \mu^{\prec}, < \rangle$ is not λ^+-superuniversal.*

PROOF. The existence of strongly admissible well orderings follows from 4.1.4. To prove that $\langle H_\kappa, \mu^{\prec}, < \rangle$ is weakly κ-superuniversal, let $\langle T, v\rangle$ be a metric space of cardinality less than κ, let f be a member of $\mathscr{B}\mathscr{S}\mathscr{I}(T,S)$, and let $\mathscr{S} = \{\gamma_\beta : \beta < \lambda\}$ be an appropriate sequence under which $\prec$ is strongly $\mathscr{S}$-admissible. Then if we choose any β such that $\mathrm{bn}(f)$, $|T| \leqq \gamma_\beta$, it follows immediately from 4.1.2 that f can be extended element by element to an isometry g from T into H_κ such that $\mathrm{bn}(g) < 2^{\gamma_\beta}$.

Part 2 now follows from 1 and 4.1.1. To prove part 3, we first note that if $T = \langle T, v\rangle$ is any λ^+-superuniversal space and U is any unitary subspace of T of cardinality λ, then there exists a point $t \in T - U$ such that

$$v(u,t) = 1 \qquad \text{for every } u \in U.$$

Thus it will be sufficient to find such a subset $U \in \mathscr{P}^+_\lambda(H_\kappa)$ for which no appropriate point $p \in H_\kappa$ exists. We shall define U inductively, and we begin by choosing some *arbitrary* strictly increasing unbounded subset $V = \{v_\alpha : \alpha < \lambda\}$ of H_κ.

We shall set $U = \{u_\alpha : \alpha < \lambda\}$, and we let $u_0 = 0$. Now assume that we have already constructed $U^\alpha = \{u_\beta : \beta < \alpha\}$ such that

$$\mu^{\prec}(u_\beta, u_\gamma) = 1 \qquad \text{for all } \beta < \gamma < \alpha.$$

We wish to choose the point u_α. Choose any point $w_\alpha \in H_\kappa$ such that

$$w_\alpha > v_\alpha \text{ and } w_\alpha > u_\beta \qquad \text{for all } \beta < \alpha,$$

and define a function $f \in \mathscr{F}_\kappa$ by

$$f(u) = 1 \qquad \text{for all } u \in U^\alpha, \text{ and}$$

$$f(w_\alpha) = \inf(\{1 + \mu^{\prec}(u, w_\alpha) : u \in U\}) = 1 + \inf(\{\mu^{\prec}(u, w_\alpha) : u \in U^\alpha\}).$$

The function f is clearly consistent and not superfluous, so it must appear as f^{w_γ} in the construction of $H^{\prec}_\kappa$ for some $\gamma > w_\alpha$. Set $u_\alpha = \gamma$.

We see immediately that U is an unbounded subset of H_κ of cardinality λ and that, for any distinct $u, v \in U$, we have $\mu^{\prec}(u,v) = 1$. Now suppose $t \in H_\kappa - U$. Since U is unbounded, there exists an α such that $t < u_\alpha$. However, it is easily seen that

$$\mu^{\prec}(t, u_\alpha) = \inf(\{1 + \mu^{\prec}(t,u) : u \in U^\alpha\}) = 1 + \inf(\{\mu^{\prec}(t,u) : u \in U^\alpha\}),$$

and it is also clear that $\inf(\{\mu^{\prec}(t,u): u \in U^{\alpha}\})$ cannot equal zero. ∎

We can use this to extend 3.4 to:

4.6 COROLLARY. *There exists a κ-superuniversal metric space of cardinality κ iff*

$$2^{\bar{\kappa}} = \kappa \text{ and } \kappa \text{ is regular.}$$

PROOF. It is immediate from 3.4 that we need only prove that, for no singular cardinals κ, do there exist κ-superuniversal spaces of cardinality κ. So suppose there is such a space T of cardinality κ where κ is singular. By 3.2 we have $2^{\bar{\kappa}} = \kappa$, and since by König's lemma, $2^{cf(\kappa)}$ cannot equal κ, κ must be strongly singular. If we now examine the proof of 4.3, we see that it will still hold if one of the spaces in question is κ-superuniversal rather than weakly κ-superuniversal. Thus T must be isometric to $H_{\kappa}^{\prec}$ and thus by 4.5.3 not even $(\mathrm{cf}(\kappa))^{+}$-superuniversal much less κ-superuniversal. ∎

We also note:

4.7 THEOREM. *If κ is any strongly singular cardinal such that $2^{\bar{\kappa}} \neq \kappa$, then there exists a weakly κ-superuniversal metric space S_{κ} of cardinality $2^{\bar{\kappa}}$ which is not isometric to any $H_{\kappa}^{\prec}$.*

PROOF. We may apply 3.7 with respect to κ^{+} (which is regular) to prove that no $H_{\kappa}^{\prec}$ contains a κ^{+}-isolated unitary space of cardinality greater than κ^{+} and then use the construction analogous to that used in 3.9 to obtain the space S_{κ} which contains a κ^{+}-isolated unitary subspace of cardinality $2^{\bar{\kappa}}$. ∎

4.8 COROLLARY. *If κ is any strongly singular cardinal, then there exists a unique (up to isometry) weakly κ-superuniversal metric space of cardinality κ iff $2^{\bar{\kappa}} = \kappa$.* ∎

We conclude this section with a study of the completions of the spaces $H_{\kappa}^{\prec}$ for strongly singular cardinals κ of cofinality $\aleph_0$. Denote the completion of the space $H_{\kappa}^{\prec}$ for such a cardinal κ by $C_{\kappa}^{\prec} = \langle C_{\kappa}^{\prec}, \mu_c^{\prec} \rangle$. We first note:

4.9 THEOREM. *If κ is any strongly singular cardinal of cofinality $\aleph_0$, and $\prec$ is any strongly admissible well ordering of $\mathscr{F}_{\kappa}$, then $C_{\kappa}^{\prec}$ has cardinality 2^{κ}.*

PROOF. Choose any κ and $\prec$ as in the hypothesis of the theorem. We shall construct a metric space S which can be embedded into $H_{\kappa}^{\prec}$ and whose completion has cardinality 2^{κ}. Let $\{\gamma_i : i \in \omega\}$ be a strictly increasing unbounded set of cardinals in κ and let

$$S = \{\langle \alpha_0, \alpha_1, \ldots, \alpha_i, \ldots \alpha_{n-1} \rangle : i < n \rightarrow \alpha_i < 2^{\gamma_i}\}.$$

Define a metric ν on S by setting

$$\nu(\langle \alpha_0, \cdots, \alpha_i, \alpha_{i+1}, \cdots, \alpha_n \rangle, \langle \alpha_0, \cdots, \alpha_i, \beta_{i+1}, \cdots, \beta_m \rangle) =$$

$$\sum_{j=i+1}^{n} (\tfrac{1}{2})^j + \sum_{j=i+1}^{m} (\tfrac{1}{2})^j \quad \text{for} \quad \alpha_{i+1} \neq \beta_{i+1},$$

and let $\boldsymbol{S} = \langle S, \nu \rangle$. We may think of S as a tree with each point at the nth level having $2^{\gamma_{n+1}}$ successors. The metric can then be regarded as being obtained by setting the distance between a point at the nth level and one of its successors as $(\frac{1}{2})^{n+1}$ and setting the distances between any other two points as the distance along the shortest path between them. Because the metric is defined so simply, it is easily seen that S can be embedded into $\boldsymbol{H}_\kappa$ even though S has cardinality $2^{\bar{\kappa}}$.

We note that each path through the tree is a Cauchy sequence, that no two paths can have the same limit, and that there are 2^{κ} paths. We leave the details to the reader. Thus $C_\kappa^{\prec}$ has at least 2^{κ} points. On the other hand, $C_\kappa^{\prec}$ has at most $(2^{\bar{\kappa}})^{\aleph_0} \leqq (2^{\kappa})^{\aleph_0} = 2^{\kappa \cdot \aleph_0} = 2^{\kappa}$ points. ■

Since 2^{κ} must have cofinality greater than κ, it follows that if $C_\kappa^{\prec}$ is weakly κ-superuniversal, then it is κ-superuniversal. In fact it has large enough cardinality to be κ^{+}-superuniversal. We next show that it is none of these.

4.10 THEOREM. *If κ is any strongly singular cardinal of cofinality $\aleph_0$ and $\prec$ is any strongly admissible well ordering of $\mathscr{F}_\kappa$, then $\boldsymbol{C}_\kappa^{\prec}$ is not $\aleph_1$-superuniversal and therefore not weakly $\aleph_1$-superuniversal.*

PROOF. Let U be the subset of H_κ which we constructed in the proof of 4.5.3. We remember that U is a countable unbounded subset of H_κ satisfying:

a. $\mu^{\prec}(u, v) = 1$ for all distinct $u, v \in U$,

b. $\alpha < u \rightarrow \exists v \in U(\mu^{\prec}(u, \alpha) = 1 + \mu^{\prec}(v, \alpha))$ for all $\alpha \in H_\kappa$ and $u \in U$.

(Part b follows from the fact that the infimums involved are all over finite sets and are therefore actually attained.) Now suppose $\boldsymbol{C}_\kappa^{\prec}$ were $\aleph_1$-superuniversal. Then as we noted in the proof of 4.5.3, we could find a point $p \in C_\kappa$ such that

$$\mu_c^{\prec}(p,u) = 1 \qquad \text{for all } u \in U.$$

But $C_\kappa^{\prec}$ is the completion of H_κ, so there would have to exist an $\alpha \in (H_\kappa - U)$ such that

$$\bar{\mu}_c^{\prec}(p,\alpha) < \tfrac{1}{2}.$$

But this implies

$$\tfrac{1}{2} < \mu_c^{\prec}(u,\alpha) = \mu^{\prec}(u,\alpha) < \tfrac{3}{2} \qquad \text{for all } u \in U$$

which in turn implies

$$\mu^{\prec}(u,\alpha) - \mu^{\prec}(v,\alpha) < 1 \qquad \text{for all } u,v \in U$$

which violates condition b. ■

5. Generalizations

In this section, applications of superuniversality to various collections of spaces are considered. We show that while the concept can be easily extended to one type of bounded space, it cannot to another. We also produce two simple countable metrizable spaces such that any notion of superuniversality which included them would lead to nonHausdorff superuniversal spaces, and we conclude with an application to graph theory. We begin, however, with a generalization in another direction.

Although not stated, we have implicitly assumed that a metric space is a *set* on which a metric is defined. Suppose now we allow the space to have for its elements a proper class. (This can be done even in *ZF*, but we shall not concern ourselves with the details as the constructions are easier to visualize in set theories which admit proper classes as objects.) We then define a **metric class** to be a class C with a metric μ defined on it and we denote the resulting structure by $\mathbf{C} = \langle C, \mu \rangle$. (Technically, a bit of care is needed here because a class cannot contain proper classes as elements, so some sort of convention is needed to handle ordered pairs of classes. One possibility is to define $\langle A, B \rangle$ to be $A \times B$ whenever A and B are proper classes.)

In class-set theory, there are two forms of the axiom of choice; one says that every *set* can be well ordered while the other says the same for every *class*. As the latter is known [2] to be strictly stronger, we state our definition of superuniveısality in a form that allows us to prove the existence of such a space using only the weaker form. Thus we define a metric class $\langle C, \mu \rangle$ to be **superuniversal** iff for every metric *space* $\langle T, v \rangle$ every isometry in $\mathscr{S}\mathscr{I}(T, C)$ can be extended to an isometry in $\mathscr{I}(T, C)$. Then, letting H denote the class of all ordinals, we have:

5.1 THEOREM. *There exists a superuniversal metric class* $\langle H, \mu^H \rangle$.

PROOF. Let $\mathscr{F}$ be the class of functions from sub*sets* of H into R^+. It is not hard to see that $\mathscr{F}$ can be well ordered with the order type of H and that this requires no stronger form of the axiom of choice than that R^+ can be well ordered. But once $\mathscr{F}$ is so well ordered, we can use the construction we used in Section 2 to obtain μ^H. ■

Our earlier results now generalize to, e.g.:

5.2 THEOREM. *If* $\langle S, \mu \rangle$ *is any superuniversal metric class, then*:

1. *Every open subset of* S *is a proper class.*

2. *There exist arbitrarily large "sets" of disjoint superuniversal metric subclasses of* S.

PROOF. The proof of 1 is essentially as in 3.2. The statement of 2 is not quite correct since it is impossible to have a "set of classes". What we mean is that if A is any nonempty set, then there is a function f from some subclass of S onto A such that for every $a \in A$ the subclass $f^{-1}(a)$ is superuniversal. The proof is similar to that of 3.11 and 3.12. Some care must be exercised because S may not be well orderable, but this can be done as long as A is a set. ■

Finally, suppose we define a (metric) class to be **standard** iff it (its class of elements) can be well ordered. It is well known that every such proper class can be well ordered with the order type of H by using the notion of set rank. Thus we have:

5.3 THEOREM. *If* $\mathbf{H} = \langle H, \mu^H \rangle$ *is superuniversal, then*:

1. *Every standard superuniversal metric class is isometric to* $\mathbf{H}$.

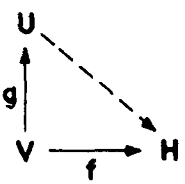
2. *If* $\langle G, v \rangle$ *is any standard metric class and* f *is any isometry from a subset of* G *into* H, *then* f *can be extended to an isometry from all of* G *into* H.

3. $\mathbf{H}$ *can be decomposed into the disjoint union of a class of superuniversal metric classes each isometric to* $\mathbf{H}$ *itself.*

PROOF. Again the proofs are essentially the same as in 3.1, 3.6, and 3.15 respectively; again 3 should be interpreted as stating the existence of a function f, this time from all of H onto H such that $f^{-1}(\alpha)$ is a superuniversal metric subclass of H which is isometric to H. ∎

We also note that the notion of superuniversity has an interesting category theoretic interpretation. Let $\mathscr{C}$ be the category of all metric classes and all metric spaces, let $\mathscr{C}^{<}$ be the category of all metric spaces, let $\mathscr{C}_\kappa$ be the category of all metric spaces of cardinality κ, let $\mathscr{C}_\kappa^{<}$ be the category of all metric spaces of cardinality less than κ, and let morphisms of all four categories be the appropriate isometries. Then we may restate 3.1.4, 4.1.2b, and their equivalents for metric classes as:

5.4 THEOREM. *If every class is standard (if $2^{\bar{\kappa}} = \kappa$) and H is any superuniversal metric class ((weakly) κ-superuniversal metric space of cardinality κ), then H is a weakly terminal member of $\mathscr{C}$ ($\mathscr{C}_\kappa$) such that if V is any member of $\mathscr{C}^{<}$ ($\mathscr{C}_\kappa^{<}$), U is any member of $\mathscr{C}$ ($\mathscr{C}_\kappa$), f is any isometry from V into H, and g is any isometry from V into U, then the diagram below can be completed so as to commute.*



∎

We next restrict the requirements of κ-superuniversality to a subclass of metric spaces. Thus for any subclass $\mathfrak{M}$ of metric spaces, we define a metric space $\langle S, \mu\rangle$ to be κ-superuniversal with respect to $\mathfrak{M}$ iff for every metric space $\langle T, \nu\rangle \in \mathfrak{M}$ of cardinality κ every isometry in $\mathscr{S}\mathscr{I}_\kappa(T,S)$ can be extended to an isometry in $\mathscr{I}(T,S)$.

In particular, we shall look at certain classes of bounded metric spaces. For any positive real number r, we define a metric space $\langle S, \mu\rangle$ to have **diameter at most r** iff

$$\mu(s,t) \leqq r \qquad \text{for all } s,t \in S,$$

and to have **diameter at most r^-** iff

$$\mu(s,t) < r \qquad \text{for all } s,t \in S.$$

We would like to find spaces which are κ-superuniversal with respect to spaces of diameter at most r which are themselves of diameter at most r and similarly for spaces of diameter at most r^-. However, while the former is easy, the latter is impossible. We have, following Urysohn:

5.5 THEOREM. *If κ is any uncountable cardinal, r is any positive real number, $\langle S, \mu \rangle$ is any κ-superuniversal metric space, and T is any subset of S of diameter at most r and cardinality less than κ, then the metric space generated by*

$$U = \{s \in S: t \in T \rightarrow \mu(s,t) = r/2\}$$

has diameter at most r and is κ-superuniversal with respect to metric spaces of diameter at most r.

PROOF. We can use essentially the same proof as in 3.11. Thus let f be any consistent function from some set $V \in \mathscr{P}_\kappa(U)$ into R^+. It is easily seen that f can be extended consistently to T by setting

$$f(t) = r/2 \qquad \text{for all } t \in T$$

and that the point then obtained by applying 3.1.6 is in U. ■

(The proof of this theorem does not generalize directly to weakly κ-superuniversal spaces because the set U might happen to be bounded in order. It is easily seen, however, that this will be no problem with the $H_\kappa^{\prec}$.) On the other hand:

5.6 THEOREM. *If r is any positive real number and **S** is any metric space which is $\aleph_1$-superuniversal for metric spaces of diameter less than r, then **S** is **not** of diameter at most r^-.*

PROOF. Let $\boldsymbol{R}$ be the real line with the usual metric and let $\boldsymbol{T}$, $\boldsymbol{T}^0$, and $\boldsymbol{T}^r$ be the subspaces generated by the sets

$$T = \{q: 0 < q < r \text{ and } q \text{ is rational}\},$$

$$T^0 = T \cup \{0\}, \text{ and } T^r = T \cup \{r\}.$$

Since $\boldsymbol{T}$ has diameter at most r^- and is countable, there exists, by the $\aleph_1$-superuniversality of $\boldsymbol{S}$, an isometry f from $\boldsymbol{T}$ into $\boldsymbol{S}$. Then, again by $\aleph_1$-superuniversality, f can be extended to isometries f^0 and f^r which take $\boldsymbol{T}^0$ and $\boldsymbol{T}^r$ into $\boldsymbol{S}$. But it is easily seen that in $\boldsymbol{S}$, the distance between $f^0(0)$ and $f^r(r)$ must be r. ■

It would also be of interest to extend these notions to topological spaces other than metric spaces. It is not clear, however, what we should require f to be. If we require that f be an embedding, then all of our constructions which proceed point by point will fail because even a countable union of embeddings need not itself be an embedding. For example, let N be the space consisting of the positive integers with the discrete topology, let f be the function from N into R defined by

$$f(1) = 0$$

$$f(n+1) = 1/n \qquad \text{for all } n \geqq 1,$$

and for each positive integer n let

$$f_n = f \restriction \{m : m \leqq n\}.$$

It is easily seen that each f_n is an embedding of its domain into R, but f is not an embedding.

If, on the other hand, we merely require that f be a continuous injection, then it is easy to find a space of cardinality κ which is superuniversal with respect to all topological spaces of cardinality κ, namely, the indiscrete space of cardinality κ. Ideally, of course, we would prefer to find superuniversal spaces with the same separation properties as the spaces they correspond to (e.g., a normal space which is super universal with respect to normal spaces), but, as we shall see, this will be impossible for almost anything stronger than T_1 spaces.

More precisely, suppose $\mathfrak{A}$ is a class of topological structures, κ is a cardinal, and S is a topological space. Then we define S to be **κ-superuniversal for $\mathfrak{A}$** iff for every space $\langle T, \mathcal{O} \rangle \in \mathfrak{A}$ of cardinality less than κ and every continuous injection f from a subset $U \subseteq T$ into S, there exists a continuous injection g which extends f and has domain T.

We define A and B to be the subspaces of R with the inherited topology generated by the sets

$$A = \{1/n : n \in N\} \cup \{0\}, \text{ and}$$

$$B = A \cup \{2 + 1/n : n \in N\} \cup \{2\}$$

and we prove:

5.7 THEOREM. *If $\mathfrak{A}$ is any class of topological spaces containing A and B, and S is any space which is $\aleph_1$-superuniversal for $\mathfrak{A}$, then S is not Hausdorff.*

PROOF. Let f be any continuous injection from A into S, and let g be the continuous injection from $B - \{2\}$ into S defined by

$$g(0) = 0,$$
$$g(1/n) = f(1/2n), \quad \text{and}$$
$$g(2 + 1/n) = f(1/(2n+1)).$$

Then by the $\aleph_1$-superuniversality of S, g can be extended to a countinuous injection from B into S. It is easily seen, however, that $g(0)$ and $g(1)$ cannot be separated by open sets in S. ■

It should be noted that both A and B are both very simple spaces. They are countable, compact, metrizable, and have at most two accumulation points.

We conclude this section with an application to graph theory. We note that the construction in Section 2 can be modified to obtain all kinds of κ-superuniversal graphs (directed graphs, graphs in which points may be connected by many edges, graphs in which points may be connected to themselves, etc.) and that these new constructions will be far simpler than required for metric spaces because we need not worry about either consistency or superfluity. (For a treatment of the countable case and some generalization to higher cardinals, see [8].) However, in many cases such κ-superuniversal graphs can be constructed directly from a κ-superuniversal space. For example, suppose we define a **Michigan graph** to be a pair $\langle V, E\rangle$ where V (the set of **vertices**) is some non-empty set and E (the set of **edges**) is any subset of $\mathscr{P}_2^+(V) - \mathscr{P}_1^+(V)$. Thus a Michigan graph is one in which no vertex is connected to itself and any two vertices are connected by at most one edge. For any vertex $v \in V$, we define the **link** of v by setting

$$\ln(v) = \{u : \{u, v\} \in E\},$$

the **colink** of v to be

$$\operatorname{cln}(v) = \{u \neq v : u \in V - \ln(v)\},$$

and the **degree** and **codegree** of v to be $|\ln(v)|$ and $|\operatorname{cln}(v)|$ respectively. Finally, we define the **complement** of $\langle V, E\rangle$ to be the graph $\langle V, \mathscr{P}_2^+(V) - (\mathscr{P}_1^+(V) \cup E)\rangle$. Using these, we mention without proof:

5.8 Theorem. *If κ is any infinite cardinal, then:*

1. *If $\langle V, E\rangle$ is any κ-superuniversal Michigan graph, then each vertex has degree and codegree $2^{\bar{\kappa}}$, and the link and colink of each vertex are themselves κ-superuniversal as is the complement of the graph.*

2. If $\langle S, \mu\rangle$ *is any κ-superuniversal metric space, then the graph*

$$\langle S, \{\{u,v\} : \mu(u,v) = 1\}\rangle$$

is a κ-superuniversal Michigan graph. ■

5.9. COROLLARY *If κ is any regular infinite cardinal, then the smallest κ-superuniversal Michigan graph has cardinality $2^{\bar{\kappa}}$.*

PROOF. The result follows immediately except that the case $\kappa = \aleph_0$ must be handled separately using either Uryshon's construction or the obvious modification of the constructions in Section 2. ■

Many other theorems in Section 3 also generalize immediately to κ-superuniversal graphs, and 5.8.1 with slight modifications holds even if κ is finite, e.g., the links and colinks are $(\kappa - 1)$-superuniversal. We shall treat this entire subject in greater detail elsewhere.

6. Open problems

In this section we mention some open problems connected with our earlier sections.

1. Can there exist a metric space of cardinality less than $2^{\bar{\kappa}}$ which is universal for all metric spaces of cardinality κ? In particular, does there always exist such a space of cardinality κ?

2. Can superuniversal spaces be characterized by their topological properties?

3. Does every (weakly) κ-superuniversal metric space contain a unitary subspace of cardinality $2^{\bar{\kappa}}$?

4. Can every κ-superuniversal metric space be decomposed into a disjoint union of κ-superuniversal subspaces?

5. Does every κ-superuniversal metric space contain a proper isometric subspace?

6. If κ is regular and $2^{\bar{\kappa}} \neq \kappa$, then:

1. Do there exist two κ-superuniversal spaces of cardinality $2^{\bar{\kappa}}$ which are not only not isometric but also not homeomorphic?
2. Do there exist admissible well orderings $\prec$ and $\prec^*$ of $\mathscr{F}_\kappa$ such that $N_\kappa^{\prec}$ and $N_n^{\prec *}$ are not homeomorphic or even not isometric?
3. Does each $H_\kappa^{\prec}$ contain a proper isometric subspace?
4. Can each $H_\kappa^{\prec}$ be decomposed into a disjoint union of $2^{\bar{\kappa}}$ or fewer κ-superuniversal subspaces?

7. For a given cardinal κ, do there exist κ-superuniversal spaces of all cardinalities greater than $2^{\bar{\kappa}}$? In particular, if the generalized continuum hypothesis holds, then by using methods to construct the $H_{\aleph_1}^{<}$, we can easily construct $\aleph_1$-superuniversal spaces of every uncountable cardinality less than $\aleph_\omega$. Is there one of cardinality exactly $\aleph_\omega$?

8. If p is any point in any topological space, then there is a cardinal κ, which we shall call the **local density** of p, such that some open neighborhood V of p has cardinality κ and every other neighborhood of p contained in V also has κ points. Does there exist a κ-superuniversal space with two points of differing local density?

9. How can κ-superuniversal spaces be combined to create "larger" such spaces? In particular, is the product of two such spaces κ-superuniversal?

10. If κ is a strongly singular cardinal, then how large is the smallest κ-superuniversal metric space? In particular, if $2^{\bar{\kappa}} \neq \kappa$, could there be such a space of cardinality $2^{\bar{\kappa}}$?

11. Are there κ-superuniversal spaces which are not weakly κ-superuniversal?

12. In 4.1.2, is the hypothesis $\mathrm{cf}(|S|) \geqq \mathrm{cf}(\kappa)$ really needed?

13. Do 4.7 and 4.9 hold for the completions of arbitrary weakly κ-superuniversal spaces of cardinality $2^{\bar{\kappa}}$?

14. Does 5.5 hold for all weakly κ-superuniversal spaces?

15. Is it consistent that there exist a superuniversal metric class which is not standard? If so:

1. Need it contain a unitary class?
2. Need it contain a "class" of disjoint superuniversal subclasses?
3. Need it be decomposible into a union of superuniversal subclasses?
4. Need it contain a standard superuniversal subclass?
5. Need it satisfy 5.3.2?

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